1.Explain the principle and characteristics of a matched filter. Hence derive the expression for its frequency response function.

Matched-Filter Receiver:

A network whose frequency-response function maximizes the output peak-signal-to-mean-noise (power) ratio is called a matched filter. This criterion, or its equivalent, is used for the design of almost all radar receivers.

The frequency-response function, denoted H (f), expresses the relative amplitude and phase of the output of a network with respect to the input when the input is a pure sinusoid. The magnitude |H (f)| of the frequency-response function is the receiver amplitude passband characteristic. If the bandwidth of the receiver passband is wide compared with that occupied by the signal energy, extraneous noise is introduced by the excess bandwidth which lowers the output signal-to-noise ratio. On the other hand, if the receiver bandwidth is narrower than the bandwidth occupied by the signal, the noise energy is reduced along with a considerable part of the signal energy. The net result is again a lowered signal-to-noise ratio. Thus there is an optimum bandwidth at which the signal-to-noise ratio is a maximum. This is well known to the radar receiver designer. The rule of thumb quoted in pulse radar practice is that the receiver bandwidth B should be approximately equal to the reciprocal of the pulse width τ. This is a reasonable approximation for pulse radars with conventional superheterodyne receivers. It is not generally valid for other waveforms, however, and is mentioned to illustrate in a qualitative manner the effect of the receiver characteristic on signal-to-noise ratio. The exact specification of the optimum receiver characteristic involves the frequency-response function and the shape of the received waveform.

The receiver frequency-response function, is assumed to apply from the antenna terminals to the output of the IF amplifier. (The second detector and video portion of the well-designed radar superheterodyne receiver will have negligible effect on the output signal-to-noise ratio if the receiver is designed as a matched filter.) Narrow banding is most conveniently accomplished in the IF. The bandwidths of the RF and mixer stages of the normal superheterodyne receiver are usually large compared with the IF bandwidth. Therefore the frequency-response function of the portion of the receiver included between the antenna terminals to the output of the IF amplifier is taken to be that of the IF amplifier alone. Thus we need only obtain the frequency-response function that maximizes the signal-to-noise ratio at the output of the IF. The IF amplifier may be considered as a filter with gain. The response of this filter as a function of frequency is the property of interest. For a received waveform s(t) with a given ratio of signal energy E to noise energy N_o (or noise power per hertz of bandwidth), North showed that the frequency-response function of the linear, time-invariant filter which maximizes the output peak-signal-to-mean-noise (power) ratio for a fixed input signal-to-noise (energy) ratio is

$$H(f) = G_a S^*(f) \exp(-j2\pi f t_1)$$

where $S(f) = \int_{-\infty}^{\infty} s(t) \exp(-j2\pi f t)\, dt$ = voltage spectrum (Fourier transform) of input signal

$S^*(f)$ = complex conjugate of $S(f)$

t_1 = fixed value of time at which signal is observed to be maximum

G_a = constant equal to maximum filter gain (generally taken to be unity)

The noise that accompanies the signal is assumed to be stationary and to have a uniform spectrum (white noise). It need not be gaussian. The filter whose frequency-response function is given by Eq. above has been called the North filter, the conjugate filter, or more usually the matched filter. It has also been called the Fourier transform criterion. It should not be confused with the circuit-theory concept of impedance matching, which maximizes the power transfer rather than the signal-to-noise ratio.

The frequency-response function of the matched filter is the conjugate of the spectrum of the received waveform except for the phase shift exp (- j2Πft$_1$). This phase shift varies uniformly with frequency. Its effect is to cause a constant time delay. A time delay is necessary in the specification of the filter for reasons of physical realizability since there can be no output from the filter until the signal is applied. The frequency spectrum of the received signal may be written as an amplitude spectrum |S(f)| (and a phase spectrum exp [-jφ$_s$ (f)]. The matched- filter frequency-response function may similarly be written in terms of its amplitude and phase spectra |H (f)| and exp [-jφ$_m$ (f)]. Ignoring the constant G$_a$, Eq. above for the matched filter may then be written as

$$|H(f)| \exp[-j\phi_m(f)] = |S(f)| \exp\{j[\phi_s(f) - 2\pi f t_1]\}$$

or

$$|H(f)| = |S(f)|$$

and

$$\phi_m(f) = -\phi_s(f) + 2\pi f t_1$$

Thus the amplitude spectrum of the matched filter is the same as the amplitude spectrum of the signal, but the phase spectrum of the matched filter is the negative of the phase spectrum of the signal plus a phase shift proportional to frequency.

The matched filter may also be specified by its impulse response h (t), which is the inverse Fourier transform of the frequency-response function.

$$h(t) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi ft)\, df$$

Physically, the impulse response is the output of the filter as a function of time when the input is an impulse (delta function).

$$h(t) = G_a \int_{-\infty}^{\infty} S^*(f) \exp[-j2\pi f(t_1 - t)]\, df$$

Since $S^*(f) = S(-f)$, we have

$$h(t) = G_a \int_{-\infty}^{\infty} S(f) \exp[j2\pi f(t_1 - t)]\, df = G_a s(t_1 - t)$$

A rather interesting result is that the impulse response of the matched filter is the image of the received waveform; that is, it is the same as the received signal run backward in time starting from the fixed time t_1. Figure 1 shows a received waveform s (t) and the impulse response h (t) of its matched filter. The impulse response of the filter, if it is to be realizable, is not defined for t< 0. (One cannot have any response before the impulse is applied.) Therefore we must always have $t < t_1$. This is equivalent to the condition placed on the transfer function H (f) that there be a phase shift exp ($-j2\Pi ft_1$). However, for the sake of convenience, the impulse response of the matched filter is sometimes written simply as s (- t).

Derivation of the matched-filter characteristic: The frequency-response function of the matched filter has been derived by a number of authors using either the calculus of variations or the Schwartz inequality. We shall derive the matched-filter frequency-response function using the Schwartz inequality.

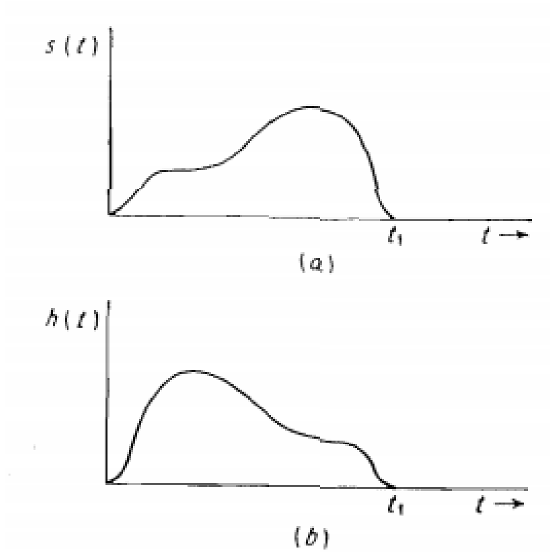

Fig.1 (a) Received waveform s(t); (b) impulse response h(t) of the matched filter.

We wish to show that the frequency-response function of the linear, time-invariant filter which maximizes the output peak-signal-to-mean-noise ratio is

$$H(f) = G_a S^*(f) \exp(-j2\pi f t_1)$$

when the input noise is stationary and white (uniform spectral density). The ratio we wish to maximize is

$$R_f = \frac{|s_o(t)|^2_{max}}{N}$$

where $|s_o(t)|_{max}$ = maximum value of output signal voltage and N = mean noise power at receiver output. The ratio R_f is not quite the same as the signal-to-noise ratio which has been considered in the radar equation. The output voltage of a filter with frequency-response function H(f) is

$$|s_o(t)| = \left| \int_{-\infty}^{\infty} S(f) H(f) \exp(j2\pi f t)\, df \right|$$

where S(f) is the Fourier transform of the input (received) signal. The mean output noise power is

$$N = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

where N_o is the input noise power per unit bandwidth. The factor appears before the integral because the limits extend from $-\infty$ to $+\infty$, whereas N_o is defined as the noise power per cycle of bandwidth over positive values only. Assuming that the maximum value of $|s_o(t)|^2$ occurs at time $t = t_1$, the ratio R_f becomes

$$R_f = \frac{\left| \int_{-\infty}^{\infty} S(f) H(f) \exp(j2\pi f t_1) \, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df}$$

Schwartz's inequality states that if P and Q are two complex functions, then

$$\int P^* P \, dx \int Q^* Q \, dx \geq \left| \int P^* Q \, dx \right|^2$$

The equality sign applies when P = kQ, where k is a constant. Letting

$$P^* = S(f) \exp(j2\pi f t_1) \quad \text{and} \quad Q = H(f)$$

and recalling that

$$\int P^* P \, dx = \int |P|^2 \, dx$$

we get, on applying the Schwartz inequality to the numerator of Eq. earlier, we get

$$R_f \leq \frac{\int_{-\infty}^{\infty} |H(f)|^2\, df \int_{-\infty}^{\infty} |S(f)|^2\, df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df} = \frac{\int_{-\infty}^{\infty} |S(f)|^2\, df}{\frac{N_0}{2}}$$

From Parseval's theorem,

$$\int_{-\infty}^{\infty} |S(f)|^2\, df = \int_{-\infty}^{\infty} s^2(t)\, dt = \text{signal energy} = E$$

Therefore we have

$$R_f \leq \frac{2E}{N_0}$$

The frequency-response function which maximizes the peak-signal-to-mean-noise ratio R_f may be obtained by noting that the equality sign in Eq. applies when P = kQ, or

$$H(f) = G_a S^*(f) \exp(-j2\pi f t_1)$$

where the constant k has been set equal to $1/G_a$.

2. Discuss the relation between the matched filter characteristics and correlation function.

The matched filter and the correlation function. The output of the matched filter is not a replica of the input signal. However, from the point of view of detecting signals in noise, preserving the shape of the signal is of no importance. If it is necessary to preserve the shape of the input pulse rather than maximize the output signal-to-noise ratio, some other criterion must be employed.

The output of the matched filter may be shown to be proportional to the input signal cross-correlated with a replica of the transmitted signal, except for the time delay t_1. The cross-correlation function R (t) of two signals $y(\lambda)$ and $s(\lambda)$, each of finite duration, is defined as

$$R(t) = \int_{-\infty}^{\infty} y(\lambda) s(\lambda - t)\, d\lambda$$

The output $y_o(t)$ of a filter with impulse response h(t) when the input is $y_{in}(t) = s(t) + n(t)$ is

$$y_0(t) = \int_{-\infty}^{\infty} y_{in}(\lambda) h(t - \lambda)\, d\lambda$$

If the filter is a matched filter, then $h(\lambda) = s(t_1 - \lambda)$ and Eq. above becomes

$$y_0(t) = \int_{-\infty}^{\infty} y_{in}(\lambda) s(t_1 - t + \lambda)\, d\lambda = R(t - t_1)$$

Thus the matched filter forms the cross correlation between the received signal corrupted by noise and a replica of the transmitted signal. The replica of the transmitted signal is "built in" to the matched filter via the frequency-response function. If the input signal $y_{in}(t)$ were the same as the signal s(t) for which the matched filter was designed (that is, the noise is assumed negligible), the output would be the autocorrelation function. The autocorrelation function of a rectangular pulse of width τ is a triangle whose base is of width 2τ.

3. Briefly explain about the efficiency of non-matched filters.

Efficiency of non matched filters: In practice the matched filter cannot always be obtained exactly. It is appropriate, therefore, to examine the efficiency of non matched filters compared with the ideal matched filter. The measure of efficiency is taken as the peak signal-to-noise ratio from the non matched filter divided by the peak signal-to-noise ratio ($2E/N_o$) from the matched filter. Figure.3.1 plots the efficiency for a single-tuned (RLC) resonant filter and a rectangular-shaped filter of half-power bandwidth B_τ when the input is a rectangular pulse of width τ. The maximum efficiency of the single-tuned filter occurs for $B_\tau \approx 0.4$. The corresponding loss in signal-to-noise ratio is 0.88 dB as compared with a matched filter. Table 3.2 lists the values of B_τ which maximize the signal-to-noise ratio (SNR) for various combinations of filters and pulse shapes. It can be seen that the loss in SNR incurred by use of these non-matched filters is small.

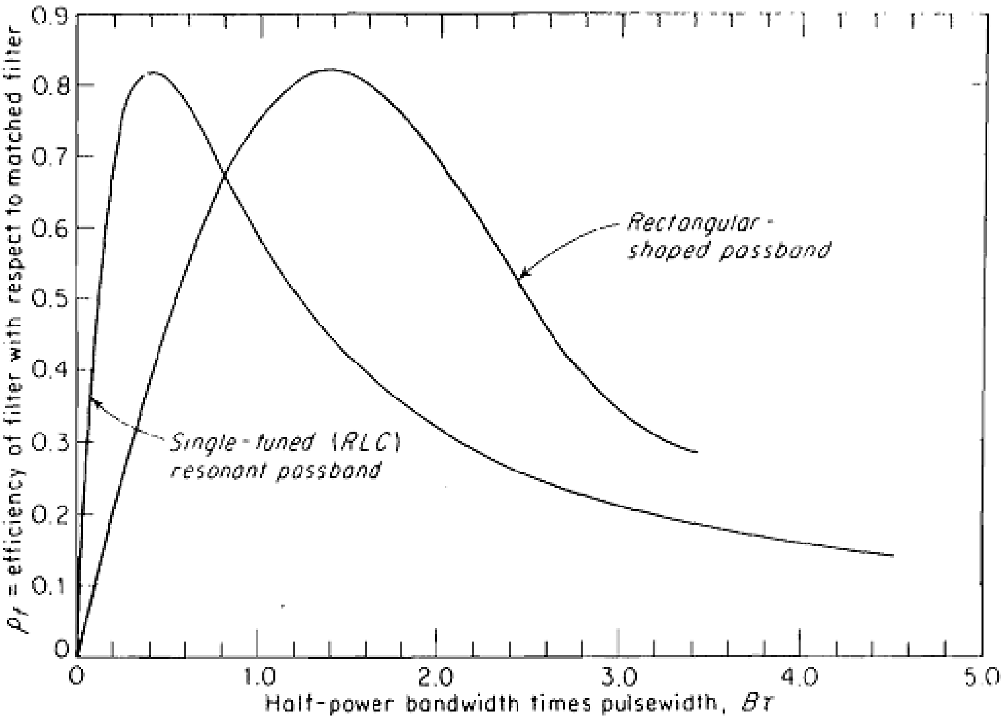

Fig.3.1 Efficiency, relative to a matched filter, of a single-tuned resonant filter and a rectangular shaped filter, when the input signal is a rectangular pulse of width τ. B = filter bandwidth.

Input signal	Filter	Optimum $B\tau$	Loss in SNR compared with matched filter, dB
Rectangular pulse	Rectangular	1.37	0.85
Rectangular pulse	Gaussian	0.72	0.49
Gaussian pulse	Rectangular	0.72	0.49
Gaussian pulse	Gaussian	0.44	0 (matched)
Rectangular pulse	One-stage, single-tuned circuit	0.4	0.88
Rectangular pulse	2 cascaded single-tuned stages	0.613	0.56
Rectangular pulse	5 cascaded single-tuned stages	0.672	0.5

Table 3.2 Efficiency of nonmatched filters compared with the matched filter

4. Derive the expression for frequency response of the matched filter with non-white noise.

Matched filter with nonwhite noise: In the derivation of the matched-filter characteristic, the spectrum of the noise accompanying the signal was assumed to be white; that is, it was independent of frequency. If this assumption were not true, the filter which maximizes the output signal-to-noise ratio would not be the same as the matched filter. It has been shown that if the input power spectrum of the interfering noise is given by $[N_i\,(f)]^2$, the frequency-response function of the filter which maximizes the output signal-to-noise ratio is

$$H(f) = \frac{G_a S^*(f) \exp(-j2\pi f t_1)}{[N_i(f)]^2}$$

When the noise is nonwhite, the filter which maximizes the output signal-to-noise ratio is called the NWN (nonwhite noise) matched filter. For white noise $[N_i(f)]^2$ = constant and the NWN matched-filler frequency-response function of Eq. above reduces to that of Eq. discussed earlier in white noise. Equation above can be written as

$$H(f) = \frac{1}{N_i(f)} \times G_a \left(\frac{S(f)}{N_i(f)} \right)^* \exp(-j2\pi f t_1)$$

This indicates that the NWN matched filter can be considered as the cascade of two filters. The first filter, with frequency-response function $1/N_i\,(f)$, acts to make the noise spectrum uniform, or white. It is sometimes called the whitening filter. The second is the matched filter when the input is white noise and a signal whose spectrum is $S(f)/N_i(f)$.

5. Explain the principle and process of correlation detection.

Correlation Detection:

$$y_0(t) = \int_{-\infty}^{\infty} y_{in}(\lambda) s(t_1 - t + \lambda)\, d\lambda = R(t - t_1)$$

Equation above describes the output of the matched filter as the cross correlation between the input signal and a delayed replica of the transmitted signal. This implies that the matched-filter receiver can be replaced by a cross-correlation receiver that performs the same mathematical operation as shown in Fig.5. The input signal y (t) is multiplied by a delayed replica of the transmitted signal $s(t - T_r)$, and the product is passed through a low-pass filter to perform the integration. The cross-correlation receiver of Fig.5 tests for the presence of a target at only a single time delay T_r. Targets at other time delays, or ranges, might be found by varying T_r. However, this requires a longer search time. The search time can be reduced by adding

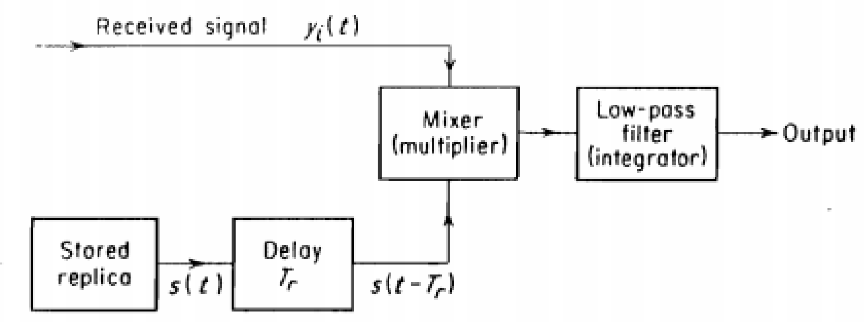

Fig.5 Block diagram of a cross-correlation receiver.

parallel channels, each containing a delay line corresponding to a particular value of T_r, as well as a multiplier and low-pass filter. In some applications it may be possible to record the signal on some storage medium, and at a higher playback speed perform the search sequentially with different values of T_r. That is, the playback speed is increased in proportion to the number of time-delay intervals T_r that are to be tested.

Since the cross-correlation receiver and the matched-filter receiver are equivalent mathematically, the choice as to which one to use in a particular radar application is determined by which is more practical to implement. The matched-filter receiver, or an approximation, has been generally preferred in the vast majority of applications.